Strategyproof Sharing of Submodular Access Costs: Budget Balance versus Efficiency

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May 1996

Abstract

A given set of users share the submodular cost of access to a network (or, more generally, the submodular cost of any idiosyncratic binary good). We compare strategyproof mechanisms that serve the efficient set of users (but do not necessarily balance the budget) with those that exactly cover costs (but are not necessarily efficient). Under the requirements of individual rationality (guaranteeing voluntary participation) and consumer sovereignty (an agent will obtain access if his willingness to pay is high enough), we find:

i) a unique strategyproof and efficient mechanism (a variant of the familiar pivotal mechanism) dubbed the marginal contribution mechanism (MC)

ii) a whole class of strategyproof and budget-balanced mechanisms, each one corresponding to a certain cost sharing formula; these mechanisms, unlike MC, are immune to manipulations by coalitions

Within the second class, the mechanism associated with the Shapley value cost sharing formula is characterized by the property that its worst welfare loss is minimal. We compare the budget imbalances of MC with the welfare losses of the Shapley value mechanism. Application of these methods to the case of a tree network without congestion is also discussed.

[0]Stimulating conversations with Raj Deb, Bhaskar Dutta, Wolfgang Pesendorfer, and William Thomson are gratefully acknowledged.

1 The Problem and the Punchline

Sharing the cost of a service jointly produced for a given set of users is one of the most popular applications of cooperative game theory (*e.g.*, Shubik [1962], Loehman and Whinston [1979], Straffin and Heaney [1989]). The standard model has a set N of users who can either have access to the service or not; if the subset (coalition) S of N receives the service, the cost $C(S)$ must be shared between the users in S. As the cost function C is not necessarily symmetrical in the different users, we can think of "service to user i" as an idiosyncratic good different from "service to user j". One canonical example of such a system, whose terminology we will use throughout the paper for convenience, is that of providing access to a network, where the network may be distributing water (Okada et al. [1982]), or telecommunication services (Sharkey [1995]), or multicast messages (Herzog et al. [1995]), and so on; we discuss this network interpretation more thoroughly in Section 7.

The literature on cost allocation (surveyed, for example, in Young [1985] and Moulin [1988,1995]) mostly addresses the two related questions of fairness (which cost allocations among users are fair?) and core stability (which cost allocations are not threatened by the secession of a coalition standing alone to provide its own service?). In this paper, we look at the somewhat different issue of incentive compatibility; we consider mechanisms that elicit from each user his/her willingness to pay for access to the network, then decide who gets access and how the cost is shared. Strategyproofness is the minimal level of incentive compatibility we require, and to that end we focus exclusively on strategyproof mechanisms, where truthful report of one's willingness to pay is always a dominant strategy. We also discuss another form of incentive compatibility, namely group strategyproofness (where no coalition of users has an incentive to jointly misreport their true willingness to pay).

There are other properties, besides incentive compatibility, that such cost allocation mechanisms should have. In particular, one would naturally like the cost allocations to be efficient (maximize the total welfare) and also budget balanced (cover costs exactly). Unfortunately, even in problems where preferences can be represented by quasi-linear utilities (as we assume in this paper), there are no strategyproof cost allocation mechanisms that are both budget balanced and efficient (Green, Kohlberg, and Laffont [1976]). In this paper we explore the efficiency/budget-balance trade off in the particular problem of access to a network (as described above). We conclude that, in this context, the strategyproof and budget balanced mechanisms are preferable to strategyproof and efficient ones because of their superior incentive compatibility properties and the greater normative flexibility they offer the mechanism designer. Accordingly, dropping the efficiency requirement is potentially more useful than dropping budget balance in the access charge problem. Before laying out our arguments, we describe the kind of context for which a choice between efficiency and budget balance makes sense.

A context where strategyproof and efficient (hence budget-imbalanced) mechanisms are often proposed (*e.g.*, Green and Laffont [1979]) is when a public authority elicits individual preferences and implements an allocation with the mandate of maximizing total welfare and the ability to absorb any deficit or surplus that may occur during the implementation of the mechanism (the public authority acts as a banker, covering the deficit or siphoning off the surplus without distorting the incentives). The corresponding set of strategyproof and efficient mechanisms under quasilinear utilities is well known; they are usually called the Clarke-Groves mechanisms (Clarke [1971], Groves [1973]).

A context where budget balance is inescapable is cooperative production (Israelsen [1980], Roemer [1986, 1989], Moulin [1987], Moulin and Shenker [1992], Fleurbaey and Maniquet [1994]); the users of the service are in autarchy, so that covering costs is a feasibility constraint. The corresponding class of strategyproof (but inefficient) mechanisms has not been previously characterized in the literature; our Theorem 1 offers such a characterization under a stronger incentive compatibility requirement than strategyproofness (*i.e.*, ruling out profitable manipulations by coalitions of agents; see Section 3).

In many realistic situations, both kinds of mechanisms (efficient and budget imbalanced; budget balanced and inefficient) are in fact available. A subset of members of a club (or a subset of division managers in a firm, or any subcoalition in a larger community) want to provide a certain service, but the service will not be offered to the membership at large: *e.g.*, the residents of a certain neighborhood want to jointly establish a local network of bicycle trails (and the existing highways prevent the extension of the network beyond this particular neighborhood). This local project may or may not be allowed to tap into the budget of the larger community; that is to say, both kinds of mechanisms are conceivable and thus a normative comparison is called for.

We cast the focus of this paper in the above context.[1] To support our conclusion that strategyproof and budget balanced mechanisms are preferable to strategyproof and efficient ones, we offer two characterization results. First, we identify a particular strategyproof and efficient mechanism (called the marginal contribution mechanism) and show that it is the only reasonable one in this class: Proposition 1 in Section 3 (the meaning of reasonableness is given at the beginning of Section 3). Thus the mechanism designer has no flexibility at all once he chooses to impose the efficiency requirement; in particular he must treat all agents equally (in the sense that if two users impose equivalent costs then equal messages must result in equal allocations). Second, we uncover a rich class of budget balanced and strategyproof mechanisms; each one of them corresponds (in a manner explained in Section 4) to a particular cost sharing formula satisfying a certain monotonicity property. This class gives the designer a fair amount of flexibility, including treating equals unequally by means of an asymmetric cost sharing formula (typically, ranking the agents in an arbitrary yet fixed fashion and charging stand alone cost to the first ranked agent requesting service, incremental cost to the second ranked agent requesting service, and so on) or treating equals equally by using an equitable formula (*e.g.*, the Shapley value). Moreover, every mechanism in this class satisfies a (much) stronger incentive compatibility requirement than strategyproofness called group strategyproofness; they are immune to joint misreports by coalitions of any size (whereas the marginal contribution mechanism is extremely vulnerable to such moves, as explained at the end of Section 3). Theorem 1 in Section 4 characterizes the class of reasonable group strategyproof and budget balanced mechanisms.

Within the above class we single out the one associated with the Shapley value formula, because it

[1] Yet another context of interest is that of a profit maximizing firm supplying access to the network by means of a strategyproof pricing mechanism. Although we do not discuss this context, it should be clear from Proposition 1 that a profit maximizing firm will not use a strategyproof and efficient pricing mechanism.

generates the smallest potential welfare loss (the smallest deviation from efficiency): Theorem 2 in Section 5. Thus the paramount equitable solution of the cost sharing problem – the Shapley value – ends up being justified on incentive grounds, without invoking any consideration of equity (not even equal treatments of equals). In Section 6 we compare the maximal welfare loss of the Shapley value mechanism with the maximal budget imbalance of the marginal contribution mechanism, and find that neither is systematically larger than the other.

The main restrictive assumption, maintained throughout most of the paper, is that the cost function C is submodular; that is, the marginal cost $C(S \cup i) - C(S)$ of adding user i to the set of users S is nonincreasing in S (*i.e.*, usage of the network by new agents can only reduce the cost of including a given agent). Submodular cost functions (also called concave cost functions, as in Shapley [1971]) have many remarkable properties pertaining to core stability (see, for example, Moulin [1988] Chapter 5). In the context of access to a network, a very natural family of submodular cost functions arise in tree networks without congestion (Sharkey [1995]). Fix a tree from the source to all potential users (each user sitting on a node of the tree); a cost is assigned to each link (or edge) of the tree. The cost of serving any coalition S is simply the sum of the costs of the links (irrespective of the number of users "downstream" of each link) needed to reach all members of S. An important recent example of such a problem is the allocation of costs in multicast transmissions on the Internet (Herzog et al. [1995]). Section 7 is devoted to the application of our two outstanding mechanisms to various tree networks.

2 The Model

Given are N, the set of users, and C, the nondecreasing, nonnegative and submodular cost function:

$$C(\emptyset) = 0 \; ; \quad S \subseteq T \Rightarrow C(S) \leq C(T)$$

$$C(S \cap T) + C(S \cup T) \leq C(S) + C(T) \; for \; any \; S, T \subseteq N$$

We denote by u_i, $u_i \geq 0$, the amount user i is willing to pay for access to the service. A revelation mechanism elicits from each user his/her willingness to pay and, based on these reports, it chooses who gets access and how the costs are shared.[2] We use the vectors x and q to denote the resulting allocation decisions; x_i is the cost share imputed to agent i, and q_i describes whether or not agent i is given access ($q_i = 0$ if agent i is not given access, and $q_i = 1$ if agent i is given access). When we want to make the utility profile dependence of these allocations explicit, we will write $q(u)$ and $x(u)$ to denote the allocations resulting at a particular profile. We assume quasi-linear utilities, so an agent's utility is simply given by $u_i q_i - x_i$. We postulate that a "reasonable" mechanism must satisfy the following three properties:

No Subsidy (NS) The cost shares are nonnegative: $x_i \geq 0$ for all i.

Voluntary Participation (VP) The welfare level corresponding to no access ($q_i = 0$) at no cost ($x_i = 0$) is guaranteed to each user if they report truthfully.

[2] Following established, but somewhat confusing, notational practice, the vector u will denote the reported values as well as the truthful values. When the two differ, we will make special note of it.

Consumer Sovereignty (CS) Every user has a message u_i guaranteeing that, regardless of the other reported values u_{-i}, he/she gets access ($q_i = 1$).

Voluntary participation is the standard individual rationality constraint; user i does not prefer the "status quo" ($q_i = x_i = 0$) allocation to the allocation (q_i, x_i) assigned by the mechanism. In view of No Subsidy, this implies that she pays nothing if she gets no service ($q_i = 0 \Rightarrow x_i = 0$).

Consumer Sovereignty means that by reporting a high enough willingness to pay, any user is guaranteed to get the service. For instance, in the mechanisms discussed below, reporting any $u_i \geq C(\{i\})$ (their stand alone cost) ensures that $q_i = 1$ no matter what the other reported u_j are. Note that we do not require that reporting $u_i = 0$ ensures that $q_i = 0$; as long as $x_i = 0$ we can have $q_i = 1$ even though $u_i = 0$. If $C(S)$ is strictly increasing in S, however, the later configuration ($u_i = 0$ but $q_i = 1$) will not happen in any of the mechanisms discussed below.[3]

The strategyproofness literature for cost sharing problems discusses two classes of mechanisms, corresponding to the choice between budget balance and efficiency. The first class are often called the Clarke-Groves (CG) mechanisms (Clarke [1971], Groves [1973]); given the profile of individual reports, such a mechanism computes an efficient coalition S^* and assigns payments x_i, $i = 1, \ldots, n$ to individual users in such a way that truthful reporting of one's willingness to pay is a dominant strategy for each user. Requiring strategyproofness and efficiency of S^* precludes having a profile of payments that exactly covers the cost of S^* for all profiles (Green, Kohlberg, and Laffont [1976]); the mechanism may run a deficit at certain profiles, namely $x_N < C(S^*)$, or a surplus at certain profiles, namely $x_N > C(S^*)$, or both, but we know it cannot be budget balanced, $x_N = C(S^*)$, at all profiles.[4]

In Section 3, we show that there is a unique CG mechanism satisfying NS and VP (Proposition 1), and that this mechanism satisfies CS as well. Denote by $w(S, u)$ (or simply $w(S)$ if the underlying utility profile is unambiguous) the stand alone surplus of coalition S given the technology C:

$$w(S, u) = \max_{T \subseteq S} [u_T - C(T)] \tag{1}$$

The unique CG mechanism (meeting NS and VP) works as follows: given the profile of reports u_i, $i \in N$, compute the largest efficient coalition S^* (*i.e.*, the largest solution of the program $w(N, u)$; submodularity of C ensures that this largest coalition is well-defined). Then set $x_i = 0$ for each $i \notin S^*$ and assign the following payment to each $i \in S^*$:

$$x_i = u_i - ((w(N, u) - w(N - i, u)) \tag{2}$$

We call it the *marginal contribution* mechanism because, if given access, agent i's net utility gain $u_i - x_i$ is exactly equal to her marginal contribution $w(N, u) - w(N - i, u)$ to the stand alone surplus function w (and if not given access then her marginal contribution is zero, $w(N, u) = w(N - i, u)$). In the case of a symmetric cost function (if all users are equivalent, $C(S)$ depends only upon the

[3] If $u_i = 0$ but $C(S + i) > C(S - i)$ for all $S \subseteq N$, then i is not part of the efficient coalition. In the mechanisms discussed below, the coalition served is always a subset of the efficient coalition, so $u_i = 0 \Rightarrow q_i = 0$ whenever agent i imposes strictly positive incremental costs.

[4] We use the standard notation that $z_S = \sum_{i \in S} z_i$.

cardinality of S, so $C(S) = c(|S|)$ for some function $c(\cdot)$) this mechanism charges the marginal cost $c(|S^*|) \Leftrightarrow c(|S^*| \Leftrightarrow 1)$ to all users. While this is quite similar in form to the traditional *pivotal* mechanism,[5] there is an important difference as well; the marginal contribution mechanism never runs a budget surplus (i.e, it either is budget balanced or runs a budget deficit), whereas the pivotal mechanism never runs a budget deficit (i.e, it either is budget balanced or runs a surplus).[6]

The second class of strategyproof mechanisms sharing the cost of jointly produced goods, insists on budget balance at all profiles but may fail to serve the efficient coalition of users; that is, it provides service to a coalition S of users that may not maximize the reported welfare ($u_S \Leftrightarrow C(S)$ may be less than $w(N, u)$).[7] As we discuss below, the strategyproof and budget balanced mechanisms we consider are derived from an underlying *cost sharing method*. A cost sharing method is a formula ξ associating to each cost function C and to each coalition S of users an allocation of the total cost $C(S)$ among these users in the form of nonnegative cost shares $\xi_i(S)$.[8] These cost shares must satisfy the budget balancedness constraint $\xi_S(S) = C(S)$.

We now explain why these cost sharing methods underlie the relevant strategyproof and budget balanced mechanisms. Given a cost sharing method and a profile u of willingness to pay, consider the (normal form) *demand game* where each user decides whether or not to request service (user i's strategy choice is $q_i = 0$ or $q_i = 1$) and costs are shared among those who request service according to the cost sharing method ξ. If it so happens that for all profiles u the demand game has a unique α-equilibrium (where α-equilibrium can be either a Nash equilibrium or a strong equilibrium) then any α-equilibrium outcome defines a strategyproof revelation mechanism. That is, this mechanism elicits u_i, $i \in N$, and implements the unique α-equilibrium outcome of the reported demand game. This general fact is a classic result of the implementation literature that hinges on the property of the preference domain known as monotonic closedness: see Dasgupta, Hammond, and Maskin [1979].[9] In the particular case of sharing the cost of a jointly produced good, the general fact admits a converse statement. If we look for *group* strategyproof mechanisms satisfying NS, VP, and CS, then the only way to obtain these is by way of a *cross monotonic* cost sharing method of which the associated demand game has a unique equilibrium at all profiles.[10] In our problem, the property of *cross monotonicity* of the cost sharing method means that user i's cost share cannot increase when the set of users expands[11]

[5]The pivotal mechanism assigns the net utility $u_i - x_i = w^*(N) - w^*(N-i)$ to user i, where $w^*(S) = \max_{T \subseteq S}[u_T - \frac{|T|}{|N|}C(T)]$ (see, for example, Green and Laffont [1979]).

[6]Moreover, the pivotal mechanism fails VP.

[7]This route has been introduced much more recently: Shenker [1992,1995], Moulin and Shenker [1992], Moulin [1994,1996], Deb and Razzolini [1995ab], Ohseto [1995], Serizawa [1994].

[8]In the language of cooperative game theory, a cost sharing method is a "value" defined for all reduced games (S, C).

[9]In general, quasi-linear utilities do not yield a monotonically closed domain of individual preferences. In our particular problem, given NS and VP, we can restrict agent i's consumption set to the union of $(0, 0)$ (no service at no cost) and the half-line $(1, x_i)$ (service at some nonnegative cost). On this choice set, quasi-linear utilities form a monotonically closed domain.

[10]Two (technically very different) formulations of this statement are in Shenker [1992,1995] (for the case where the demand q_i is a real number) and Moulin [1996] (for the case where q_i is an integer, as we consider here).

[11]In some of the literature, the cross monotonicity property is known as *population monotonicity*, such as in Sprumont [1990].

$$S \subseteq T, \quad i \in S \Rightarrow \xi_i(T) \leq \xi_i(S) \tag{3}$$

(recall that $\xi_i(S)$ is user i's cost share when S is the set of users who get access). Under cross monotonicity, the demand game has nondecreasing best reply functions (if $q_i = 1$ is a best reply when a coalition S has access, then $q_i = 1$ remains a best reply when any superset of S gets access) hence the demand game possesses a Pareto superior Nash equilibrium. This equilibrium can be computed by the simultaneous Cournot tatonnement starting at N; that is to say the equilibrium outcome $\tilde{S}(\xi, u)$ is the limit of the following inclusion monotonic sequence:

$$S_0 = N \ ; \quad S_{t+1} = \{i | u_i \geq \xi_i(S_t)\} \tag{4}$$

It turns out that the outcome $\tilde{S}(\xi, u)$ is the welfarewise-unique strong equilibrium (there may be other equilibrium q's but they produce the same welfare) of the demand game (N, ξ, u), hence the corresponding revelation mechanism (eliciting the profile u, then implementing $\tilde{S}(\xi, u)$, and sharing the cost $C(\tilde{S}(\xi, u))$ according to ξ) is strategyproof. It is group strategyproof as well. The relevant characterization here is Theorem 2 in Moulin [1996], stating that every group strategyproof revelation mechanism satisfying Budget Balance (costs are exactly covered), NS, VP, and CS obtains as above as the strong equilibrium outcome of a cross monotonic cost sharing method (this result is stated here as Theorem 1 in Section 4).

Proposition 1 (characterizing the MC mechanism) and Theorem 1 yield the first contribution of this paper; together, they show that the combination of budget balance and strategyproofness allows greater flexibility to the designer and stronger incentive properties than the combination of efficiency and strategyproofness. The former combination, in particular, allows for more flexibility than the latter; in the former case, agent i's cost share can be any value between her stand alone cost $C(\{i\})$ and her incremental cost $C(S) \Leftrightarrow C(S \Leftrightarrow i)$ (where S is the coalition of users actually served), whereas in the latter case her cost share is entirely determined. See the discussion in Section 4.

We now turn to the second contribution of the paper. Within the class of cross monotonic cost sharing methods, we show that the *Shapley value* (Shapley [1953]) is characterized by the property that the worst welfare loss of its associated revelation mechanism is minimal (Theorem 2 in Section 5). In other words, if we denote by $\gamma(\xi, u)$ the welfare loss produced by a mechanism ξ at a profile u, so

$$\gamma(\xi, u) = w(N, u) \Leftrightarrow (u_{\tilde{S}(\xi,u)} \Leftrightarrow C(\tilde{S}(\xi, u))) \tag{5}$$

then what we show is that the maximum of $\gamma(\xi, u)$ over all $u \in \Re^n_+$ is smallest when the method ξ is the Shapley value formula. Note that the Shapley value formula is cross monotonic when costs are submodular (see Sprumont [1990]), so it is indeed in the class of methods we are considering. Thus, somewhat surprisingly, this *minimax* property – minimizing the maximal welfare loss – leads to a characterization of the Shapley value (over the class of submodular cost functions) based exclusively on incentive compatibility and efficiency considerations.[12]

[12]Another characterization of the Shapley value that uses no considerations of equity is based on its potential property (*i.e.*, the fact that cost shares are the derivatives of a potential function: Hart and Mas-Colell [1989]). Actually, the proof of our Theorem 2 uses the potential property as well; see Section 5.

Finally, we devote Sections 6 and 7 to a quantitative comparison of our two outstanding strategyproof mechanisms, namely marginal contribution ((2)) and the mechanism associated with the Shapley value. Both mechanisms have a shortfall – a budget deficit for the marginal contribution mechanism and an unrealized welfare gain for the mechanism based on the Shapley value – and these can be quantitatively compared. Even though the nature of these shortfalls are quite different, the relative sizes the shortfalls incurred in our two mechanisms is still an interesting point of comparison (as discussed in Section 1). We compute explicitly (Proposition 2) the maximal welfare loss γ of the Shapley value mechanism (namely the quantity $\gamma = \sup_{u \in \Re^n_+} \gamma(\xi, u)$ where $\gamma(\xi, u)$ is defined by (5) and ξ is taken to be the Shapley value mechanism) and the maximal budget imbalance β of the marginal contribution mechanism, namely

$$\beta = \sup_{u \in \Re^n_+} [C(S^*(u)) \Leftrightarrow x_N(u)] \tag{6}$$

where $x(u)$ is given by (2). The two formulae are, respectively,

$$\beta = [\sum_{i \in N} C(N \Leftrightarrow i)] \Leftrightarrow (n \Leftrightarrow 1)C(N) \tag{7}$$

$$\gamma = \left[\sum_{S \subseteq N} \frac{(|S| \Leftrightarrow 1)!(|N| \Leftrightarrow |S|)!}{|N|!} C(S) \right] \Leftrightarrow C(N) \tag{8}$$

For any cost function C, γ is smaller than β if $n = 2$ or 3. For larger values of n, the comparison can go either way but in many simple examples γ exceeds β. In any event, the ratio $\frac{\gamma}{\beta}$ is bounded away from zero and infinity independently of the submodular cost function C (Corollary 3 to Proposition 2).

Section 7 is devoted to the special case of cost sharing of a tree network when cost is additive along the links of the network and there are no congestion effects (*i.e.*, the cost of each link is independent of the number of users of the link). We compute β and γ for a variety of simple tree structures, and confirm the findings of Section 6 that for small n, γ is smaller than β but the inequality is often reversed for larger values of n. However, certain tree structures yield $\gamma < \beta$ for any size.

We conclude in Section 8 by raising some additional issues for future consideration. These issues include the average case (rather than worst-case) analysis of shortfalls, other comparative aspects such as the computational complexity of the mechanisms, and the consideration of supermodular (rather than submodular) cost functions.

3 The Marginal Contribution Mechanism

Given a fixed cost function C, a *revelation mechanism* is a mapping M associating to all profiles $u \in \Re^n_+$ a coalition $S(u) \subseteq N$ (equivalently expressed by the binary inclusion vector $q(u)$ with $q_i(u) = 1$ for all $i \in S(u)$ and $q_i(u) = 0$ for all $i \notin S(u)$) and a vector $x(u)$ of *nonnegative* monetary compensations. Budget balance is not required, so $x_N(u) \Leftrightarrow C(S(u))$ can be nonzero at some profiles. In addition to the No Subsidy property ($x_i(u) \geq 0$ for all i, all u) we now write the other two properties introduced in Section 2 as follows:

- *Voluntary Participation* (VP): for all profiles u, all agents i

$$u_i q_i(u) \Leftrightarrow x_i(u) \geq 0$$

- *Consumer Sovereignty* (CS): for all $i \in N$ there is a value u_i such that, for all u_{-i}, $S(u)$ contains i (where u_{-i} is the $N \Leftrightarrow i$ projection of u).

We say that M is *strategyproof* if for all profiles u, all agents i, and all "misreports" u', we have

$$u_i q_i(u) \Leftrightarrow x_i(u) \geq u_i q_i(u_{-i}, u'_i) \Leftrightarrow x_i(u_{-i}, u'_i)$$

In order to define our first mechanism, we state without proof two easy and important consequences of the submodularity of C for the surplus function $w(S, u)$ and the maximizing coalitions.

Fact 1: For any u, $w(\cdot, u)$ is supermodular: $w(S,u) + w(T,u) \leq w(S \cup T, u) + w(S \cap T, u)$ for any $S, T \subseteq N$.

Fact 2: If any two coalitions S and T are efficient (*i.e.*, $u_S \Leftrightarrow C(S) = u_T \Leftrightarrow C(T) = w(N,u)$) then so is $S \cup T$; we denote by $S^*(u)$ the largest efficient coalition at u, and denote by $q^*(u)$ the corresponding inclusion vector.

The marginal contribution MC mechanism picks the coalition $S(u) = S^*(u)$ and charges the cost shares $x^*(u)$ given by formula (2).

Proposition 1 *Consider a strategyproof revelation mechanism M selecting an efficient allocation (not necessarily the largest) at all profiles, meeting NS and VP. Then M is welfare equivalent to MC: for all u, all i*

$$u_i q_i(u) \Leftrightarrow x_i(u) = u_i q_i^*(u) \Leftrightarrow x_i^*(u)$$

Conversely, the MC mechanism meets NS and VP (as well as CS) and is strategyproof.

Proof: To check that MC satisfies NS, first pick a profile u and an agent i in an efficient coalition, $i \in S^*(u)$; nonnegativity of $x_i^*(u)$ amounts to $w(N,u) \leq u_i + w(N \Leftrightarrow i, u)$, which follows from

$$w(N,u) = u_{S^*(u)} \Leftrightarrow C(S^*(u)) \leq u_i + u_{S^*(u)-i} \Leftrightarrow C(S^*(u) \Leftrightarrow i) \leq u_i + w(N \Leftrightarrow i, u)$$

VP is a straightforward consequence of $w(N \Leftrightarrow i, u) \leq w(N, u)$. CS follows from the fact (due to submodularity) that, for a given i, if $u_i \geq C(\{i\})$ then $u_{S+i} \Leftrightarrow C(S+i) \geq u_S \Leftrightarrow C(S)$ for all $S \subseteq N$ and all $u_{-i} \in \Re_+^{n-1}$.

Conversely, choose a mechanism M as in the premises of the first statement. By the standard characterization result of CG mechanisms (*e.g.*, Green and Laffont [1979] or Moulin [1988]) the property of strategyproofness combined with the efficiency of $S(u)$ implies the following form for $x(u)$:

$$x_i(u) = u_i q_i(u) - (w(N,u) - h_i(u_{-i}))$$

Consider an arbitrary $(N - i)$ profile u_{-i} and denote by u^0 its completion to an N-profile such that $u_i^0 = 0$. We must have $x_i(u^0) = 0$ (by NS and VP, as noted at the beginning of Section 2) and the above equation yields

$$h_i(u_{-i}) = w(N, u^0) = w(N - i, u^0) = w(N - i, u)$$

for any profile u with $(N - i)$ projection u_{-i}. Thus, the mechanism M takes the same form as MC, except for the fact that $S(u)$ can be any efficient coalition (not necessarily the largest one as in MC). In order to check that the two mechanisms are welfare equivalent, we need only to consider an agent i in $S^*(u) - S(u)$ because both mechanisms give the precisely same allocation to every agent in $S(u)$ or outside of $S^*(u)$. For any agent i in $S^*(u) - S(u)$, we have $w(N,u) = w(N - i, u)$ hence she receives from the two mechanisms the (q_i, x_i) allocations of $(0,0)$ and $(1, u_i)$ respectively, which are welfare equivalent. □

Manipulating the marginal contribution mechanism

Strategyproofness does not prevent all forms of manipulation of the mechanism, and indeed there are stronger forms of incentive compatibility to consider. *Group strategyproofness* expresses the property that no coalition of users will find it profitable to jointly misreport their willingness to pay. To make this precise, fix any coalition $T \subseteq N$ and any two profiles u, u' such that $u_j = u'_j$ for all $j \notin T$. Denote by (q, x) and (q', x') the allocations implemented at u and u' respectively. Group strategyproofness requires that if the inequality

$$u_i q'_i - x'_i \geq u_i q_i - x_i$$

holds for all $i \in T$ then it must be an equality for all $i \in T$ as well.

The marginal contribution method (MC) is *not* group strategyproof. To see this, consider a three agent problem with a symmetric cost function

$$C(S) = c(|S|) \text{ where } c(1) = 3,\ c(2) = 5,\ \text{and } c(3) = 6$$

The utility profile is not symmetric:

$$u_1 = 4; u_2 = u_3 = 1.2$$

Straightforward computations yield $S^*(u) = \{1\}$, $w(N,u) = 1$, $w(N - 1, u) = 0$, and $w(N - 2, u) = w(N - 3, u) = 1$. Hence, MC yields the net utility $(1,0,0)$. However, we see that the coalition $\{2,3\}$ can successfully manipulate by reporting $u'_2 = u'_3 = 4$ because at profile $u' = (4,4,4)$ we have $S^*(u') = N$, $w(N,u') = 6$, and $w(N - i, u') = 3$, so $x'_i = 4 - (6 - 3) = 1$. At this profile, agents $2,3$ end up with $q_i = 1$, $x_i = 1$, and a positive net utility; naturally, the outcome is inefficient but all agents (including agent 1) benefit from the misreport by agents $2, 3$. The banker is the loser, as he must compensate for 3 units of budget deficit.

In the above example the manipulation by coalition $\{23\}$ requires reliable coordination between these two users, because agent 1 would suffer a net loss if she were to report $u'_2 = 4$ while agent 3 is still reporting truthfully (he would end up paying 1.8 for the service).

However, there is a special kind of manipulation (again a violation of group strategyproofness) that does not require any such coordination between agents and to which the MC mechanism is always vulnerable. Fix a profile u (the true profile) and a willingness to pay u'_i for agent i such that $u_i < u'_i$. Denoting $u'_i = (u_i, u'_i)$ we observe the following facts:[13]

Fact 1: $S^*(u) \subseteq S^*(u')$

Fact 2: no agent j, $j \neq i$, is hurt: $w(N, u) \Leftrightarrow w(N \Leftrightarrow j, u) \leq w(N, u') \Leftrightarrow w(N \Leftrightarrow j, u')$

Fact 3: if $i \in S^*(u)$ then $S^*(u') = S^*(u)$ and $x_i^*(u) = x_i^*(u')$

Facts 2 and 3 tell us that if an agent i is served at profile u, reporting a higher willingness to pay is a matter of indifference to her, and it can never hurt any other agent. To see that in some cases it can strictly help other agents, consider the following example with two agents:

$$C(1) = C(2) = 6\ ; \quad C(12) = 8\ ; \quad u_1 = u_2 = 5$$

Truthful reports yield $S^*(u) = \{12\}$ and cost shares $x_1^*(u) = x_2^*(u) = 3$. By raising his willingness to pay to $u'_1 = 7$, agent 1 does not affect his allocation and reduces agent 2's cost share to $x_2^*(u'_1, u_2) = 2$. Once again, the banker is the sucker!

Since MC is the unique CG mechanism that satisfies NS and VP, there is no way to prevent manipulation by coalitions without abandoning the goal of efficiency. We now turn to the class of mechanisms that do just this.

4 Cross Monotonic Cost Sharing Methods

Recall that a cost sharing method is a mapping ξ associating to each coalition S a vector $(\xi_i(S), i \in S)$ of *nonnegative* cost shares such that $\xi_S(S) = C(S)$. Assuming that the method ξ is cross monotonic (property (3)) then the algorithm defined in formula (4) defines an inclusion monotonic

[13]To check Fact 1, denote $S = S^*(u)$, $T = S^*(u')$ and compute

$$u_{S\cap T} - C(S\cap T) \leq u_S - C(S) \Rightarrow C(S\cup T) - C(T) \leq C(S) - C(S\cap T) \leq u_{S-T} \leq u'_{S-T} \Rightarrow u'_T - C(T) \leq u'_{S\cup T} - C(S\cup T)$$

implying, by definition of T, $S \cup T = T$. Similar reasoning establishes $S^*(u, N-j) \subseteq S^*(u, N)$ for all j. To prove Fact 2, namely $w(N-j, u') - w(N-j, u) \leq w(N, u') - w(N, u)$, we note that both sides of the inequality are nonnegative and we distinguish three cases. If $S^*(N-j, u)$ contains i, so do $S^*(N, u)$, $S^*(N, u')$, and $S^*(N-j, u')$ and both terms of the inequality equal $u'_i - u_i$. If $S^*(N-j, u')$ does not contain i, neither does $S^*(N-j, u)$ and the left hand term is zero. If $i \notin S^*(N-j, u)$ but $i \in S^*(N-j, u')$, call u_i^* the smallest value such that $S^*(N-j, (u_{-i}, u_i^*))$ contains i. Because $S^*(N, (u_{-i}, u_i^*))$ contains i as well, the righthand term in the inequality is at least $u'_i - u_i^*$ whereas the lefthand term equals $u'_i - u_i^*$. The easy proof of Fact 3 is omitted.

sequence in N, with limit $\tilde{S}(\xi, u)$ (or $\tilde{S}(u)$ if, from the context, the underlying cost sharing method is unambiguous). We associate to the cost sharing method ξ the following revelation mechanism denoted by $RM(\xi)$:

$$S(u) = \tilde{S}(\xi, u); q_i(u) = 1 \textit{ and } x_i(u) = \xi_i(\tilde{S}(\xi, u)) \textit{ if } i \in \tilde{S}(\xi, u);\ q_i(u) = 0 \textit{ and } x_i(u) = 0 \textit{ otherwise} \tag{9}$$

The property of group strategyproofness has been defined in the previous section (after the proof of Proposition 1).

Theorem 1 *For any cross monotonic cost sharing method ξ, the mechanism $RM(\xi)$ (defined by (9)) is budget balanced, meets NS, VP, CS, and is group strategyproof. Conversely, pick a revelation mechanism M satisfying budget balance, NS, VP, CS, and group strategyproofness. Then there exists a cross monotonic cost sharing method ξ such that $RM(\xi)$ is welfare equivalent to M.*

The proposition is identical to Theorem 2 in Moulin [1996].[14] In the special case of an excludable public good ($C(S) = 1$ for all nonempty S), Deb and Razzolini [1995a,b] and Ohseto [1995] provide similar yet different characterization results.

We do not provide a proof of the "converse" statement in Proposition 1; however, the "direct" statement is easy to prove and we repeat its proof here.

Proof of direct portion of Theorem 1: We fix a cross monotonic method ξ and check that $RM(\xi)$ is group stategyproof (all other statements are trivial).

Let u be the (true) profile of willingness to pay and $S = \tilde{S}(\xi, u)$. Assume a certain coalition T manipulates at u by u' (where $u_j = u'_j$ for all $j \notin T$) and set $S' = \tilde{S}(\xi, u')$. Finally, set $S'' = S \cup S'$ and denote by q (respectively, q' and q'') the vector of 0 and 1 corresponding to S (respectively, S' and S''). Check first

$$for\ all\ i \in T:\ u_i q''_i \Leftrightarrow \xi_i(S'') \geq u_i q'_i \Leftrightarrow \xi_i(S') \tag{10}$$

This is clear if $i \notin S''$ (implying $i \notin S'$). If $i \in S'$ this follows from $S' \subseteq S''$ and cross monotonicity. Lastly if $i \in S'' \Leftrightarrow S'$ (implying $i \in S$) then $u_i \Leftrightarrow \xi_i(S) = 0$ because the manipulation property requires $u_i q'_i \Leftrightarrow \xi_i(S') \geq u_i q_i \Leftrightarrow \xi_i(S)$ and so, by cross monotonicity, $u_i \Leftrightarrow \xi_i(S'') \geq 0$ as desired.

Combined with the manipulation assumption, formula (10) implies

$$for\ all\ i \in T:\ u_i q''_i \Leftrightarrow \xi_i(S'') \geq u_i q_i \Leftrightarrow \xi_i(S) \tag{11}$$

with at least one strict inequality. We claim that (11) holds true as well for $i \notin T$. This is clear if $i \in S$ (implying $i \in S''$) by cross monotonicity, or if $i \notin S''$ (in which case both sides of the

[14] In that paper, individual preferences are "classical", namely monotonic (strictly in money), continuous and convex; this domain is generally larger than that allowed by quasi-linear utilities. However, in our problem, NS and VP imply that $x_i = 0$ if $q_i = 0$ hence agent i's consumption set is the union of $(0, 0)$ ($q_i = 0$, $x_i = 0$) and of the half line $(1, x_i)$, $x_i \geq 0$. On this set, a classical preference is entirely described by a simple number u_i (the willingness to pay) such that $(0, 0)$ is indifferent to $(1, u_i)$. Hence the classical domain and that of preferences represented by quasi-linear utilities coincide.

inequality are zero); if $i \in S'' \Leftrightarrow S$ (implying $i \in S'$) observe that $u_i \Leftrightarrow \xi_i(S') \geq 0$ (because q' is Nash equilibrium at u' and $u'_i = u_i$) so by cross monotonicity $u_i \Leftrightarrow \xi_i(S'') \geq 0$ as desired.

We have proven that (11) holds for all i with at least one strict inequality. Therefore $S'' \Leftrightarrow S$ is nonempty (or (11) would be an equality for all i) and there is a largest integer t such that $S'' \subseteq S_t$ (where the sequence S_t is derived by (4) at u). Pick an agent i in $S'' \cap (S_t \Leftrightarrow S_{t+1})$; we have

$$u_i < \xi_i(S_t) \leq \xi_i(S'') \Rightarrow u_i q''_i \Leftrightarrow \xi_i(S'') < 0$$

But this contradicts (11) as q is a Nash equilibrium at u. □

The family of revelation mechanisms uncovered by Theorem 1 leaves a fair degree of flexibility to the mechanism designer. Suppose first the agents have equal right to the use of the technology C, so that the property "equal treatment of equals" is deemed desirable: if the function C is symmetrical in i and j (*i.e.*, $C(S+i) = C(S+j)$ for all $S \subseteq N \Leftrightarrow \{i,j\}$), then the cost shares of agents i and j should be symmetrical as well.[15] Then the designer can choose from a variety of formulae, including the Shapley value (Shapley [1953])[16], the egalitarian solution (Dutta and Ray [1989])[17], convex combinations of these, and more.[18] For instance, in the case of two users $N = \{12\}$, these two values are

Shapley $\xi_1(12) = \frac{1}{2}(C(12) + C(1) \Leftrightarrow C(2))$; $\xi_2(\{1,2\}) = \frac{1}{2}(C(12) + C(2) \Leftrightarrow C(1))$

Egalitarian $\xi_1(12) = med\{\frac{C(12)}{2}, C(1), C(12) \Leftrightarrow C(2)\}$; $\xi_2(12) = med\{\frac{C(12)}{2}, C(2), C(12) \Leftrightarrow C(1)\}$

(where $med\{\cdot\}$ is the median operator).

Another possibility is that the users have unequal claims on the technology C (perhaps because of seniority, earlier contributions to its development, and so on) and that the designer wishes to choose a cost sharing formula that fairly reflects this fact. For simplicity, suppose that the users are ordered as $\{1, 2, \ldots, n\}$ and that their claims are lexicographic: agent 1's claim outweighs any other claim; agent 2's claim outweighs those of any agent $3, 4, \ldots, n$, and so on. A consequence of Theorem 1 is that this hierarchy of claims determines entirely the choice of our designer (within the class defined by NS, VP, CS, and group strategyproofness). Indeed, in any cross monotonic cost sharing method, agent 1 must pay out at least $C(S) \Leftrightarrow C(S \Leftrightarrow 1)$ where S is the coalition of users who get the service; this follows from $\xi_{S-1}(S) \leq \xi_{S-1}(S \Leftrightarrow 1) = C(S \Leftrightarrow 1)$ and $\xi_1(S) = C(S) \Leftrightarrow \xi_{S-1}(S)$. Thus, agent 1's superior claim entitles him to this minimal cost share $C(S) \Leftrightarrow C(S \Leftrightarrow 1)$. Similarly, agent 2's cost share must be at least $C(S \Leftrightarrow 1) \Leftrightarrow C(S \Leftrightarrow \{1,2\})$ (by repeating the above argument), and so on. The resulting cost sharing method is the familiar *hierarchical stand alone* method (often called the marginal contribution method) where the agent i with the largest index in S (*i.e.*, the agent with the least claim to the resources) pays her stand alone cost $C(\{i\})$, the agent

[15] Note that is is a strong version of the general idea that cost shares should reflect the inherent symmetries of the cost structure.

[16] For a proof that this value is cross monotonic when C is submodular, see, for example, Sprumont [1990].

[17] The egalitarian solution of the cooperative game (N, c) picks a profile of cost shares x^* within the stand alone core of this game ($x^*_S \leq C(S)$).

[18] See Sprumont [1990] for a systematic discussion of the cross monotonicity property. Note that the nucleolus (Schmeidler [1969]), another familiar value for cooperative games, is not cross monotonic (see Somnez [1993]).

with the next largest index j pays $C(\{ij\}) \Leftrightarrow C(\{j\})$, and so on. This method is cross monotonic given a submodular cost function (recall that the Shapley value is but the arithmetic average of such methods over all possible ordering of N).

5 The Shapley Value Mechanism

We turn to the outstanding position of the Shapley value within the class of cross monotonic cost sharing methods. Given such a method ξ, we denote by $\gamma(\xi)$ the maximal welfare loss of the mechanism $RM(\xi)$ defined by (9);

$$\gamma(\xi) = \sup_{u \in \Re^n_+} \gamma(\xi, u) = \sup_{u \in \Re^n_+} [w(N, u) \Leftrightarrow (u_{\tilde{S}(\xi,u)} \Leftrightarrow C(\tilde{S}(\xi, u)))]$$

We write as ξ^* the Shapley value cost sharing method

$$\xi^*_i(S) = \sum_{T \subseteq S-i} \frac{|T|!(|S| \Leftrightarrow |T| \Leftrightarrow 1)!}{|S|!}[C(T \cup i) \Leftrightarrow C(T)] \quad for\ all\ S\ and\ i \in S$$

We refer to the corresponding revelation mechanism $RM(\xi^*)$ as SH.

Theorem 2 [19] *Among all mechanisms $RM(\xi)$ derived from cross monotonic cost sharing methods, SH has the uniquely smallest maximal efficiency loss:*

$$\gamma(\xi^*) < \gamma(\xi) \ for\ all\ \xi \neq \xi^*$$

Proof: The proof is divided into 5 steps. In the first three steps we assume a given cross monotonic cost sharing method ξ and a utility profile u. We write $S = \tilde{S}(\xi, u)$ and $T = S^*(u)$.

Step 1: $S \subseteq T$

Assume, to the contrary, that $S \Leftrightarrow T$ is nonempty. Observe that for all $i \in S$, we have $u_i \geq \xi_i(S)$ (by construction of $\tilde{S}$ as in (4)). Hence, we compute:

$$(u_{S \cup T} \Leftrightarrow C(S \cup T)) \Leftrightarrow (u_T \Leftrightarrow C(T)) = u_{S-T} + C(T) \Leftrightarrow C(S \cup T) \geq \xi_{S-T}(S) + C(T) \Leftrightarrow \xi_{S \cup T}(S \cup T)$$

$$= (\xi_{S-T}(S) \Leftrightarrow \xi_{S-T}(S \cup T)) + (C(T) \Leftrightarrow \xi_T(S \cup T)) \geq 0$$

But this is a contradiction because T is the largest efficient coalition.

The inclusion just established implies in particular the following expression for the efficiency loss:

$$\gamma(\xi, u) = w(N, u) \Leftrightarrow w(\tilde{S}(\xi, u), u)$$

[19] Monderer and Shapley [1993] establish a seemingly unrelated property of the demand game associated with an arbitrary cost function C (not necessarily submodular, not even nondecreasing) and an arbitrary cost sharing method. They show that the demand game is a "potential game" if and only if the method is the Shapley value. Interestingly, our proof relies on the representation of the Shapley value by a *potential* function due to Hart and Mas-Colell [1989].

To see this, apply Step 1 to the restriction of the game to $\tilde{S}(\xi, u)$.

Step 2: *For all* $R \subseteq N$, $R \cap S = \emptyset$, *there exists* $i \in R$ *such that* $u_i < \xi_i(S \cup R)$

This follows from the definition of S; if we have for all $i \in R$, $\xi_i(S \cup R) \leq u_i$ then every coalition S_t in the sequence defined by (4) must contain $S \cup R$ (by cross monotonicity and an obvious induction argument), hence a contradiction.

Step 3: *Bounding* $\gamma(\xi, u)$

We represent an ordering of $N = \{1, 2, 3, \ldots, n\}$ by a permutation σ of N, with the interpretation that $\sigma(1)$ is the first in the ordering, $\sigma(2)$ is second, and so forth. We write $E(k, \sigma) = \{i | \sigma^{-1}(i) \geq k\}$ the set of users ranked k or higher by the permutation σ. We set:

$$\delta(\xi) = \max_{\sigma} \left(\sum_{i=1}^{n} \xi_{\sigma(i)}(E(i, \sigma)) \right) \Leftrightarrow C(N) \tag{12}$$

where the maximum is taken over all permutations of N.

We now show that $\gamma(\xi, u) \leq \delta(\xi)$ for the arbitrary profile u fixed at the beginning of the proof. If $S = T$, we have $\gamma(\xi, u) = 0$ and the inequality follows from the nonnegativity of δ (a consequence of cross monotonicity). From now on, assume $S \neq T$ and so, by Step 1, we must have $S \subset T$. We label (arbitrarily) the agents in $N \Leftrightarrow T$ as $\{1, 2, \ldots, k\}$ and use cross monotonicity to see that $C(N) \Leftrightarrow C(T) \leq \xi_{N-T}(N)$ and:

$$C(N) \Leftrightarrow C(T) \leq \sum_{i=1}^{k} \xi_i(N \Leftrightarrow \{1, \ldots, i \Leftrightarrow 1\})$$

Next we label the elements in $T \Leftrightarrow S$ such that $T \Leftrightarrow S = \{k+1, k+2, \ldots, k+k'\}$. We apply Step 2 repeatedly, first to $R_1 = T \Leftrightarrow S$ to pick an agent labeled $k+1$ in R_1, then to $R_2 = T \Leftrightarrow (S \cup \{k+1\})$ to pick an agent $k+2$ in R_2, and so on, with agent $k+i$ selected in $R_i = T \Leftrightarrow (S \cup \{k+1, \ldots, k+i \Leftrightarrow 1\})$ such that

$$u_{k+1} < \xi_{k+1}(T), \ldots, u_{k+i} < \xi_{k+i}(T \Leftrightarrow \{k+1, \ldots, k+i \Leftrightarrow 1\}), \ldots, u_{k+k'} < \xi_{k+k'}(S \cup \{k+k'\})$$

Finally, we label (arbitrarily) the agents in S as $\{k+k'+1, \ldots, n\}$ and repeatedly use cross monotonicity to show:

$$C(S) \leq \sum_{i=k+k'+1}^{n} \xi_i(S \Leftrightarrow \{k+k'+1, \ldots, i \Leftrightarrow 1\})$$

Summing the above inequalities yields

$$C(N) \Leftrightarrow C(T) + u_{T-S} + C(S) \leq \sum_{i=1}^{n} \xi_{\sigma(i)}(E(i, \sigma))$$

where σ is the ordering $\sigma(i) = i$, $i = 1, \ldots, n$. The lefthand term in the inequality above equals

$$(u_T \Leftrightarrow C(T)) \Leftrightarrow (u_S \Leftrightarrow C(S)) + C(N) = \gamma(\xi, u) + C(N)$$

hence the desired conclusion $\gamma(\xi, u) \leq \delta(\xi)$ follows.

Step 4: $\sup_{u \in \Re^n_+} \gamma(\xi, u) = \delta(\xi)$

In view of Step 3, it is enough to show that for any ordering σ of N, we have

$$\sup_{u \in \Re^n_+} \gamma(\xi, u) \geq \delta(\xi, \sigma) = \sum_{i=1}^{n} \xi_i(E(i, \sigma)) \Leftrightarrow C(N)$$

We pick an arbitrary ordering σ, denoted for simplicity $\sigma(i) = i$, and a number ϵ, $0 < \epsilon < 1$. Consider the utility profile

$$u_i = (1 \Leftrightarrow \epsilon)\xi_i(E(i, \sigma)) \quad \textit{for all } i \tag{13}$$

Denote by S_t the inclusion monotonic sequence (4). Note that $\tilde{S}(\xi, u)$ is typically empty; it will be empty if $\xi_i(S)$ is positive for all S. However, if some of the cost shares $\xi_i(S)$ are zero, we may have $\tilde{S}(\xi, u) \neq \emptyset$.

We now prove $\gamma(\xi, u) \geq \delta(\xi, \sigma)(1 \Leftrightarrow \epsilon) \Leftrightarrow \epsilon C(N)$. As the choice of σ and ϵ were arbitrary this will conclude the proof of Step 4. We shall use the following fact: for all S, all $i \in S$

$$\xi_i(S) = 0 \Rightarrow [C(S) = C(S \Leftrightarrow i) \textit{ and } \xi_j(S) = \xi_j(S \Leftrightarrow i) \textit{ for all } j \in S \Leftrightarrow i] \tag{14}$$

To see this, sum up the inequalities $\xi_j(S) \leq \xi_j(S \Leftrightarrow i)$ and invoke the monotonicity of the cost function.

If $\tilde{S} = \emptyset$ we have

$$\gamma(\xi, u) = w(N, u) \geq u_N \Leftrightarrow C(N) = \delta(\xi, \sigma)(1 \Leftrightarrow \epsilon) \Leftrightarrow \epsilon C(N)$$

and so we are done. However, as we noted above, $\tilde{S}$ need not be empty. We now show that this does not alter our conclusion. Define $A = \{i \in N | i \in S_i\}$, where the S_i are defined in (4). If $A = \emptyset$, then $\tilde{S} = \emptyset$ and we are done. Suppose $A \neq \emptyset$ and let i be its smallest element. For all j smaller than i we have $j \notin S_j$ hence $j \notin S_{i-1}$; thus, $S_{i-1} \subseteq E(i, \sigma)$ and the assumption $i \in S_i$ implies

$$(1 \Leftrightarrow \epsilon)\xi_i(E(i, \sigma)) = u_i \geq \xi_i(S_{i-1}) \geq \xi_i(E(i, \sigma))$$

Hence $u_i = 0$ and $\xi_i(E(i, \sigma)) = 0$. Now we can "eliminate" user i and consider the reduced problem over $N \Leftrightarrow i$ with the ordering induced by σ; let σ_{-i} denote this induced ordering. In view of (14), we have $\delta(N, \xi) = \delta(N \Leftrightarrow i, \xi)$ (noting $\xi_i(N)$ must be zero, hence $C(N) = C(N \Leftrightarrow i)$). Similarly, denoting by u_{-i} the $N \Leftrightarrow i$ profile obtained by removing u_i, we have, by (14):

$$\textit{for all } j < i, \quad \xi_i(E(j, \sigma)) = 0 \Rightarrow \xi_j(E(j, \sigma)) = \xi_j(E(j, \sigma_{-i}))$$

where $E(j, \sigma_{-i})$ are the successors of j in $N \Leftrightarrow i$. Therefore, the profile u_{-i} is defined in exactly the same way (namely by property (13)) in the reduced problem as u was in the original problem.

Consider again the sequence $S_t(\xi, u)$ defined in (4). Dropping the dependence on ξ and u temporarily, we denote by $S_t(T)$ the t'th term in the sequence defined by the program (4) with the starting point $S_0(T) = T$. Note that in general $T \subseteq T' \Rightarrow S_t(T) \subseteq S_t(T')$; thus, denoting the limit of these sequences by $\tilde{S}(T')$ and $\tilde{S}(T)$, we have $\tilde{S}(T') \neq \tilde{S}(T) \Leftrightarrow T' \Leftrightarrow T \cap \tilde{S}(T') \neq \emptyset$.

Applying this, and (14), to the two sequences $S_t(N)$ and $S_t(N \Leftrightarrow i)$, we see that either $\tilde{S}(N) = \tilde{S}(N \Leftrightarrow i)$ or $\tilde{S}(N) = \tilde{S}(N \Leftrightarrow i) \cup \{i\}$. Therefore

$$w(N, u) = w(N \Leftrightarrow i, u_{-i}); w(\tilde{S}(\xi, u), u) = w(\tilde{S}(\xi, u_{-i}), u_{-i})$$

implying $\gamma(\xi, u) = \gamma(\xi, u_{-i})$. We can now repeat the above argument for the reduced problem, and use an obvious induction argument to arrive at a reduced problem where indeed $A = \emptyset$. This completes the proof of Step 4.

Step 5:

We have proven $\gamma(\xi) = \delta(\xi)$. It remains to show the inequality $\delta(\xi^*) < \delta(\xi)$ when ξ^* is the Shapley value and ξ is another cost sharing method. First, one checks easily:

$$\frac{1}{n!}\sum_{\sigma} \delta(\xi, \sigma) = \left(\sum_{S \subseteq N} \frac{(|S| \Leftrightarrow 1)!(n \Leftrightarrow |S|)!}{n!} C(S) \right) \Leftrightarrow C(N) \tag{15}$$

where the lefthand sum is taken over all permutations of N. Note that the righthand sum is the potential function $P(N)$ of Hart and Mas-Colell [1989] who proved that the Shapley value method ξ^* can be written as:

$$\xi_i^*(S) = P(S) \Leftrightarrow P(S \Leftrightarrow i) \textit{ for all } i, \textit{ all } S, \textit{ with } P(S) = \sum_{T \subseteq S} \frac{(|T| \Leftrightarrow 1)!(|S| \Leftrightarrow |T|)!}{|S|!} C(T) \tag{16}$$

We show below, by induction on $|N|$, that the Shapley value is characterized by the property that $\delta(\xi^*, \sigma)$ is independent of σ. In view of (15) this will imply the desired conclusion, namely

$$\delta(\xi^*) = P(N) \Leftrightarrow C(N) \textit{ and } \delta(\xi) > P(N) \Leftrightarrow C(N) \textit{ for all } \xi, \ \xi \neq \xi^*$$

This claim is obvious for $n = 1$ (there is only one cost sharing method) and very easy to check for $n = 2$. Assume it holds for all N' such that $|N'| \leq n \Leftrightarrow 1$ and consider N such that $|N| = n$. Fix an ordering σ such that $\sigma(1) = i$, and denote by σ_{-i} the induced ordering on $N \Leftrightarrow i$.

Check first that $\delta(\xi^*, \sigma)$ is independent of σ by computing:

$$\delta(\xi^*, \sigma) = \xi_i^*(N) + \delta(\xi^*, \sigma_{-i}) + (C(N \Leftrightarrow i) \Leftrightarrow C(N))$$

$$= (P(N) \Leftrightarrow P(N \Leftrightarrow i)) + (P(N \Leftrightarrow i) \Leftrightarrow C(N \Leftrightarrow i)) + (C(N \Leftrightarrow i) \Leftrightarrow C(N)) = P(N) \Leftrightarrow C(N)$$

Conversely, assume that ξ is a method such that $\delta(\xi, \sigma)$ is independent of σ. Check that $\delta(\xi, \sigma_{-i})$ is independent of σ_{-i} (this is obvious from the definition of $\delta(\xi, \sigma)$), therefore, for any given σ such that $\sigma(1) = i$, we compute:

$$P(N) \Leftrightarrow C(N) = \delta(\xi, \sigma) = \xi_i(N) + \delta(\xi, \sigma_{-i}) + (C(N \Leftrightarrow i) \Leftrightarrow C(N)) = \xi_i(N) + P(N \Leftrightarrow i) \Leftrightarrow C(N)$$

so $\xi_i(N) = P(N) \Leftrightarrow P(N \Leftrightarrow i)$. The conclusion $\xi = \xi^*$ then follows from the potential formula for the Shapley value (16). $\square$

6 The Maximal Losses of MC and SH

The maximal budget deficit of MC, defined by (6), can be computed with the help of (2) which defines the cost allocations of MC:

$$\beta = \sup_{u \in \Re_+^n} \left[(n \Leftrightarrow 1) w(N, u) \Leftrightarrow \sum_{i \in N} w(N \Leftrightarrow i, u) \right] \tag{17}$$

On the other hand, the maximal welfare loss of the Shapley value mechanism SH is defined as

$$\gamma = \sup_{u \in \Re_+^n} \left[w(N, u) \Leftrightarrow (u_{\tilde{S}(\xi^*, u)} \Leftrightarrow C(\tilde{S}(\xi^*, u))) \right]$$

where $\tilde{S}(\xi^*, u)$ is the equilibrium demand computed according to (4).

Proposition 2 *For any nondecreasing and submodular cost function C such that $C(\emptyset) = 0$, we have:*

$$\beta = \left(\sum_{i \in N} C(N \Leftrightarrow i) \right) \Leftrightarrow (n \Leftrightarrow 1) C(N) \tag{18}$$

$$\gamma = P(N) \Leftrightarrow C(N) = \left(\sum_{S \subseteq N} \frac{(|S| \Leftrightarrow 1)!(n \Leftrightarrow |S|)!}{n!} C(S) \right) \Leftrightarrow C(N) \tag{19}$$

Proof: The formula (19) giving γ has been established in Step 5 of the proof of Theorem 2. We now prove (18). Check first a consequence of submodularity: the quantity $\sum_{i \in S} C(S \Leftrightarrow i) \Leftrightarrow (n \Leftrightarrow 1) C(S)$ is nondecreasing in S (using the convention that $S \Leftrightarrow i = S$ if $i \notin S$). We omit the straightforward proof. Then fix a profile u and set $S = S^*(u)$. Compute

$$(n \Leftrightarrow 1) w(N, u) \Leftrightarrow \sum_{i \in N} w(N \Leftrightarrow i, u) \leq (n \Leftrightarrow 1)(u_S \Leftrightarrow C(S)) \Leftrightarrow \sum_{i \in N} (u_{S-i} \Leftrightarrow C(S \Leftrightarrow i))$$

$$= \sum_{i \in N} C(S \Leftrightarrow i) \Leftrightarrow (n \Leftrightarrow 1) C(S) \leq \sum_{i \in N} C(N \Leftrightarrow i) \Leftrightarrow (n \Leftrightarrow 1) C(N)$$

In view of (17) this shows $\beta \leq \sum_{i \in N} C(N \Leftrightarrow i) \Leftrightarrow (n \Leftrightarrow 1) C(N)$. The converse inequality follows by choosing a profile with each u_i large enough so that $S^*(u) = N$ and $S^*(u_{-i}) = N \Leftrightarrow i$ for all i. This completes the proof. □

We derive four corollaries of Proposition 2. Corollary 1 says that with 2 or 3 users, γ never exceeds β; SH has a smaller shortfall than MC for these small systems. Corollary 2 gives an intuitive formula for γ and β in the case of symmetric cost functions. Corollary 3 computes the largest and smallest ratios $\frac{\beta}{\gamma}$; while the maximal budget imbalance of MC can be as much as n times the maximal welfare loss of SH, the latter can only be as much as $\log n$ times the former. Finally, Corollary 4 computes the worst ratio of loss to total cost, for both mechanisms; it can be as large as $\log n$ for SH but never exceeds 1 for MC.

Corollary 1 *In two agent problems ($|N| = 2$), we have*

$$\gamma = \frac{\beta}{2} = \frac{1}{2}(C(1) + C(2) \Leftrightarrow C(12))$$

In three agent problems ($|N| = 3$), we have

$$\gamma = \frac{s_1}{3} + \frac{s_2}{6} \Leftrightarrow \frac{2s_3}{3} \leq s_2 \Leftrightarrow 2s_3 = \beta \tag{20}$$

where $s_t = \sum_{S \subseteq N; |S|=t} C(S)$

Corollary 2 *If the cost function takes the form $C(S) = c(|S|)$ where c is concave on $\{0, 1, 2, \ldots, n\}$ and $c(0) = 0$, then*

$$\frac{\beta}{n} = \left(\frac{c(n)}{n} \Leftrightarrow (c(n) \Leftrightarrow c(n \Leftrightarrow 1)) \right) \tag{21}$$

$$\frac{\gamma}{n} = \frac{1}{n} \left(\sum_{i=1}^{n} \frac{c(i)}{i} \right) \Leftrightarrow \frac{c(n)}{n} \tag{22}$$

Corollary 3 *Given n, the number of users, we have*

$$\sup_C \frac{\beta(C)}{\gamma(C)} = n \ ; \quad \sup_C \frac{\gamma(C)}{\beta(C)} = f(n)$$

where the supremum is taken over all nondecreasing submodular cost functions such that $C(\emptyset) = 0$, where we use the convention that $\frac{0}{0} = 1$, and where f is the function $f(n) \equiv \sum_{i=2}^{n} \frac{1}{i}$. In particular, $\gamma(C) = 0$ if and only if $\beta(C) = 0$.

Corollary 4 *Given n, the number of users, we have*

$$\sup_C \frac{\beta(C)}{C(N)} = 1 \ ; \quad \sup_C \frac{\gamma(C)}{C(N)} = f(n)$$

(with the same notational conventions as in Corollary 3).

Proof of the Corollaries:

Corollary 1: The formulae for β, γ for the cases $n = 2, 3$ follow at once from Proposition 2. Inequality (20) amounts to $2s_1 + 8s_3 \leq 5s_2$ and follows from the combination of $s_2 \geq \frac{3}{2}s_3 + \frac{1}{2}s_1$ and $s_1 \geq s_2$ (both of which are implied by submodularity).

Corollary 2: This follows directly from the formulae for β and γ (equations (18) and (19) respectively).

Corollary 3: Note that both $\beta(C)$ and $\gamma(C)$ are symmetrical and additive functions of C; if C^σ obtains from C by permuting the agents according to σ, we have $\beta(C^\sigma) = \beta(C)$ and $\gamma(C^\sigma) = \gamma(C)$.

Therefore, if $\tilde{C} = \frac{1}{n!}\Sigma_\sigma C^\sigma$ is the cost function obtained by symmetrizing C, we have $\frac{\beta(\tilde{C})}{\gamma(\tilde{C})} = \frac{\beta(C)}{\gamma(C)}$ as well. Thus the suprema in the statement of Corollary 3 can be computed by restricting attention to symmetric cost functions where $C(S) = c(|S|)$ as in Corollary 2. In view of Corollary 2 we are left with the task of computing the bounds of the ratio

$$\frac{\sum_{i=1}^{n} \frac{c(i)}{i} \Leftrightarrow c(n)}{nc(n \Leftrightarrow 1) \Leftrightarrow (n \Leftrightarrow 1)c(n)} \tag{23}$$

over all nondecreasing concave functions c on $\{0, 1, \ldots, n\}$ such that $c(0) = 0$.

Fixing $c(n)$ and $c(n \Leftrightarrow 1)$ such that the denominator in (23) is positive, we reach the supremum of (23) by taking $c(i) = c(n) \Leftrightarrow (n \Leftrightarrow i)(c(n) \Leftrightarrow c(n \Leftrightarrow 1))$ for $i = 1, \ldots, n \Leftrightarrow 2$. Then the ratio (23) is computed to be $f(n)$. On the other hand, the minimum of (23) is reached by taking $c(i) = \frac{i}{n-1}c(n \Leftrightarrow 1)$, $i = 1, \ldots, n \Leftrightarrow 2$, and for this choice the ratio (23) is computed to be $\frac{1}{n}$.

To conclude the proof it remains to observe that the function c is linear if and only if $\beta(\check{C}) = 0$, and if and only if $\gamma(\tilde{C}) = 0$, where $\tilde{C}$ is the symmetrized form of C.

Corollary 4: To prove that $\sup_C \frac{\beta(C)}{C(N)} = 1$, note that $\beta(C) \leq C(N)$ holds by monotonicity of C. Then consider the cost function $C^*(S) = 1$ for all $S \neq \emptyset$, and observe that $\beta(C^*) = C^*(N)$. To prove that $\sup_C \frac{\gamma(C)}{C(N)} = f(n)$, note that $\gamma(C) \leq f(n)C(N)$ follows from the upper bounds on $\frac{\gamma(C)}{\beta(C)}$ and $\frac{\beta(C)}{C(N)}$ computed above. Checking the inequality $\gamma(C^*) = f(n)C^*(N)$ is straightforward.

7 Application to Cost Allocation on a Tree Network

A number of cost allocation problems in the transportation and communication industries are commonly modeled with the help of a tree network (Sharkey [1995] provides an excellent survey of the relevant game theoretic literature). Some number of users are located at the nodes of a fixed distribution tree. In the communications context, which we will use as our guide in this section, the origin of this tree is interpreted as a source sending a signal to the users; access to the network represents the ability to read the signal, be it cable TV, or Internet services, or some other medium.

In the most common model, the cost of using a given link (an edge connecting two nodes of the tree) is independent of the number of users sharing this link,[20] and the total cost of operating a certain tree is the sum of the cost of its links. Therefore, the minimal cost of serving an arbitrary coalition of users is a submodular function because the addition of new users reduces (or leaves unchanged) the set of links needed to graft a given user onto the distribution tree.[21]

[20]This represents an assumption of no *congestion* effects. This is an accurate model when looking at multicast transmissions (Herzog et al. [1995]), the construction of networks (where the expense is in laying the fiber, not in the amount of fiber), and many other examples. It is not a good model for congestion-prone facilities.

[21]We do not consider the option of changing the design of the distribution tree when members are not included; that is, denying service to a member never results in utilizing an additional link. In our model, the set of links needed to service a particular agent is fixed, and the inclusion of other members only affects what other links are needed. Minimal cost spanning trees are an example of a network where this condition fails (*e.g.*, adding a member

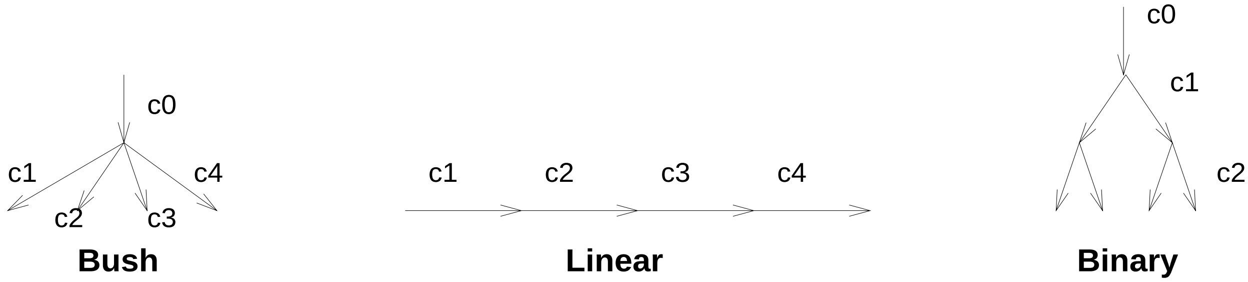

Figure 1: Three kinds of trees, each illustrated with $n = 4$.

We now compare the maximal losses β and γ of our two mechanisms MC and SH for a number of simple spanning trees. A general formula for β obtains at once. Consider a tree $\mathcal{T}$ described by the function $\mathcal{T}(N)$, where $\mathcal{T}(S)$ is the set of links needed to service the coalition S. Let $\mathcal{D}$ be the set of links followed by only one user; $\mathcal{D} = \cup_{i \in N}(\mathcal{T}(N) \Leftrightarrow \mathcal{T}(N \Leftrightarrow i))$. Then we have $\beta = C(\mathcal{T}) \Leftrightarrow C(\mathcal{D})$. This general formula is applied in the next result to three simple kinds of trees. The function $f(n) = \sum_{i=2}^{n} \frac{1}{i}$ introduced in the previous section plays a central role in the computation of γ in these simple trees. Note that $\frac{f(n)}{\log n}$ converges to 1 as n grows large, and that the first ten values of f are as follows:

n	2	3	4	5	6	7	8	9	10
$f(n)$	.5	.833	1.083	1.283	1.450	1.593	1.718	1.829	1.929

Table 1: The value of $f(n)$ for $n \leq 10$ (three significant figures).

Consider three trees depicted in Figure 7, a "bush" tree, a "linear" tree, and a "binary" tree.

Proposition 3 *For the bush tree we have*

$$\beta = c_0 \; ; \quad \gamma = f(n)c_0$$

For the linear tree we have

$$\beta = \sum_{i=1}^{n-1} c_i \; ; \quad \gamma = \sum_{i=2}^{n} f(i)c_{n-i+1}$$

For the binary tree of depth l, so $n = 2^l$, we have

$$\beta = \sum_{k=0}^{l-1} 2^k c_k \; ; \quad \gamma = \sum_{k=0}^{l-1} 2^k f(2^{l-k})c_k$$

Proof: To prove this, we need only check the computations of γ. Recall from Step 5 in the proof of Theorem 2 that for an arbitrary ordering σ of N we have

$$\gamma = \sum_{i=1}^{n} \xi_{\sigma(i)}(E(i,\sigma)) \Leftrightarrow C(N) \tag{24}$$

may result in some link no longer being needed) and may result in a non submodular cost function: see Sharkey [1995] page 724.

Pick an arbitrary link of the tree from node a to node b and an ordering of N in which all the descendants of b appear last. The cost of the link ab is shared between all the descendants of b; if there are p such descendants that cost appears with the factor $f(p)$ in the righthand sum of (24). This technique yields easily the formulae for γ in Proposition 3. □

Note that the argument in the proof also gives more general formulae for γ. Consider a tree with l levels and a splitting factor of k_i at level i and link cost c_i.

$$\gamma = \sum_{i=1}^{l-1} c_i \left(\prod_{j=1}^{i} k_j \right) f(\prod_{j=i+1}^{l} k_j)$$

For this same example, we have

$$\beta = \sum_{i=1}^{l-1} C_i \left(\prod_{j=1}^{i} k_j \right) = C(N) \Leftrightarrow nC_l$$

Note that the maximal profit (i.e., minimal revenue imbalance) arises when $l = 1$ and $k_1 = n$. Thus, maximizing profit leads to no sharing between agents, because sharing makes it harder to extract revenue from the agents.

Proposition 3 confirms the general fact that for small numbers of users γ is smaller than β, but the comparison is reversed as the number of users grows. For instance, in the bush tree, the linear tree, and the binary tree with constant link cost, one checks that $\frac{\gamma}{\beta}$ increases with n; this ratio exceeds 1 for $n \geq 4$ in the case of the bush tree, for $n \geq 6$ in the linear and binary trees. In the bush and linear trees, the ratio grows asymptotically as $\log n$; in the binary tree the ratio goes to a constant ≈ 1.118.

8 Concluding Comments

We now briefly address two additional aspects of our problem.

8.1 Supermodular Cost Functions

Another relevant context is when the cost function is supermodular rather than submodular. This occurs, for example, in tree networks with concave congestion (*i.e.*, the cost of the link is a concave function of the number of users "downstream"). For supermodular costs, the MC mechanism remains the canonical CG mechanism, as Proposition 1 and its proof remain valid word for word. However, the computation of its maximal budget imbalance becomes hard (and the imbalance can be a surplus or a deficit). On the other hand, the demand game associated with the Shapley value cost sharing method does not, in general, have a unique strong equilibrium (the best reply functions are decreasing, so multiple noncomparable equilibria are possible), hence it does not result in a strategyproof revelation mechanism. The (small) class of cost sharing methods that do result in a strategyproof mechanisms (because their demand games always have a unique strong equilibrium) has been characterized in Moulin [1996]. It reduces essentially to the family of hierarchical stand alone methods discussed in Section 4, where users pay their incremental costs according to a fixed ordering of N: $\xi_i(N) = C(E(\sigma, i)) \Leftrightarrow C(E(\sigma, i \Leftrightarrow 1))$ for some ordering σ. The criterion of minimizing the maximal welfare loss does not seem to select a clearly identifiable method in this family. Thus, in the supermodular case, the combination of group strategyproofness and budget balance proscribes equal treatment of equals, whereas the combination of strategyproofness and efficiency (as in the submodular case) prescribes equal treatment of equals. While the former (group strategyproofness and budget balance) still has superior incentive properties in the supermodular case, it no longer has claim to greater flexibility.

8.2 Incentives for Adopting Alternative Technologies

The shortfalls in the two candidate mechanisms, SH and MC, lead to interesting incentive questions in the choice of technology. Assume that the facility has committed to using either MC or SH. What happens when a facility currently using a technology with cost function C is given the chance to adopt a better technology $\tilde{C}$, where $\tilde{C}(S) \leq C(S)$ for all $S \subseteq N$.

Consider the case where the cost functions C and $\tilde{C}$ are both symmetric. Then $C(S) = c(|S|)$ and $\tilde{C}(S) = \tilde{c}(|S|)$ as in Corollary 2.

In the SH mechanism, the budget is always balanced, and so the comparison between technologies rests strictly on the welfare comparison. The Shapley cost shares take the simple form $\xi_i^*(S) = \frac{c(|S|)}{|S|}$ (and $\tilde{\xi}_i^*(S) = \frac{\tilde{c}(|S|)}{|S|}$). For any given utility profile u, the resulting equilibrium with the new technology will have higher (or equal) welfare compared with the welfare associated with the old technology because all cost shares are lower with the new technology, and so the newer equilibrium will be at least as large (and so have at least as great total willingness-to-pay, and will have lower costs). Thus, with the SH mechanisms, the incentives point unambiguously towards adoption of cheaper technologies (in the symmetric case).

In the MC mechanism, the answer is less clear. For any given utility profile u, the welfare is at least as great under the new technology. However, the budget imbalance may be larger with the new technology. Consider, for example, the case where $\tilde{c}(s) = c(s)$ for all $s < n$ and $\tilde{c}(n) = \tilde{c}(n \Leftrightarrow 1) = c(n \Leftrightarrow 1)$. The revenue R raised by the MC mechanism is given by $R = |S^*|(c(|S^*|) \Leftrightarrow c(|S^*| \Leftrightarrow 1))$ (where S^* is the efficient coalition). Then, for any u such that $S^* = N$ in the old technology, the budget loss (cost minus revenue) is higher under the new technology, with the difference being $(1 \Leftrightarrow n)(c(n) \Leftrightarrow c(n \Leftrightarrow 1))$. When looking only at the loss at a given utility profile, this perverse incentive to not adopt cheaper technologies can even occur when the new technology has merely scaled down costs, $\tilde{c}(s) = \lambda c(s)$ for some $0 < \lambda < 1$. However, if one compares instead the worst case budget imbalance for a given technology, then this perverse incentive does not occur when the costs are simply scaled down (since β scales with C) but can occur when the functional form of $\tilde{c}$ exhibits much smaller marginal costs (proportional to the total cost) of including additional users. Thus, technologies such as multicast transmissions which do not change the stand alone costs but rather increase the extent to which the resource is shared, may lead to increased rather than decreased losses.

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